Escape the Matrix: Graphical Reasoning and Minimal Axioms for Quantum Circuits

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PhD Day Edition 2025

May 6, 2025

Outline

Motivation: The Matrix Problem

What happens if you try to verify a *20-gate* quantum circuit by hand?

- You multiply 20 matrices, each $2^n \times 2^n$, by hand
- Errors multiply and it becomes intractable for even 5 qubits

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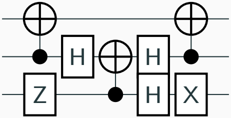
How graphical reasoning turns that mountain of matrices into friendly diagrams



- Rewrite rules on wires & nodes
- Local transformations, no full-blown matrix multiplication
- Fully automated in tools like `Quantomatic`, `PyZX`, ...

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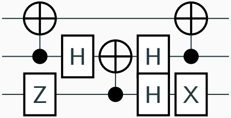

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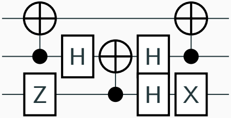

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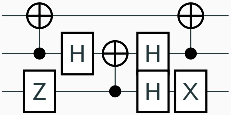
Circuit Optimization with Rewrites

Goal: Simplify a complex quantum circuit while preserving its function.

Before:



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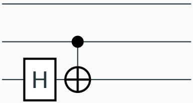

Such rewrites come from a set of *equational rules* (axioms).

Equational Theory: Musical Analogy

- **Generators:** Musical note durations (whole, half, quarter, ...)
- **Equations:** Duration identities, e.g. half $+$ half $=$ whole
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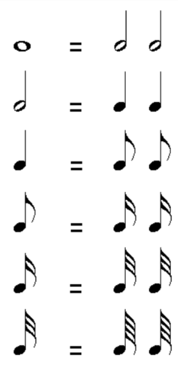
Note Duration Example

Original

half note + half note

Rewritten

whole note



A small, minimal rule delivers powerful rewriting capabilities.

What is Completeness?

A set of circuit equations is **complete** if:

- *Any* two circuits implementing the same unitary can be transformed into each other by these rules.
- Guarantees no missing identities: every true equality is derivable.

Why it matters: Ensures we can automate verification of equivalence.

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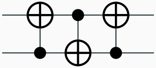
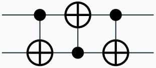
Completeness Example: SWAP Circuits

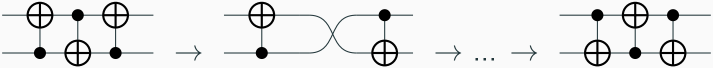
Two implementations of a SWAP gate:



(3 CNOTs)



A complete theory provides a sequence of rewrites:



proving semantic equivalence without matrices.

What is Minimality?

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Proving Independence via Interpretations

- To show axiom X is necessary:
  - Construct an *alternate interpretation* where all other axioms hold,
  - But axiom X fails (gives different semantics).
- This counter-model demonstrates X cannot be derived from the rest.

Example: Interpretation which counts the parity of some generators.

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Qubit Clifford & Real Clifford

- **Qubit Clifford** : Core operations. Widely used for error correction and efficient classical simulation. It uses phases $\pm 1, \pm i$ (complex roots).
- **Real Clifford** : Same as Qubit Clifford but with only real-valued phases ± 1, akin to working over $\mathbb{Z}_2$ instead of $\mathbb{Z}_4$.

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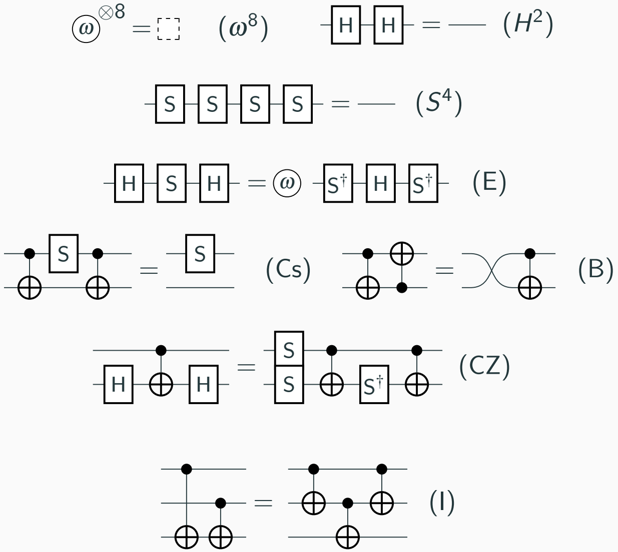
Equational Theory for Qubit Clifford Circuits



Example: Necessity of H^2 Axiom

Interpretation: Checks the presence of H gates in a circuit

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$$\bot \, \textcircled{\omega}^{\otimes 8} = \boxed{} \bot \; (\omega^8)$$

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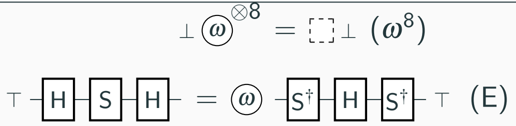

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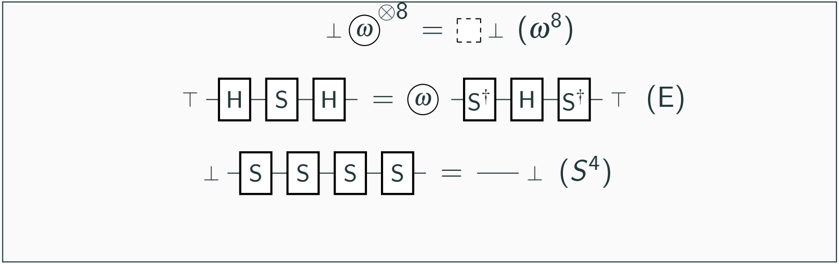

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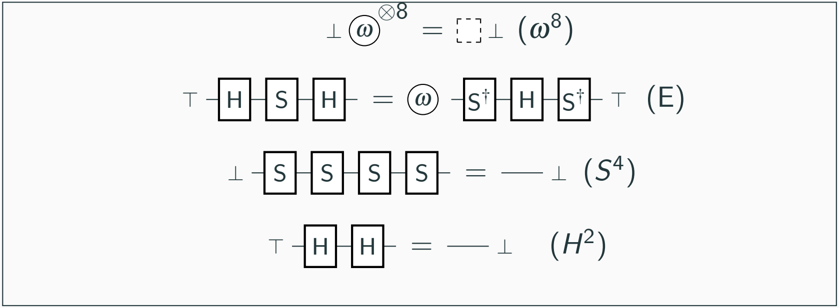

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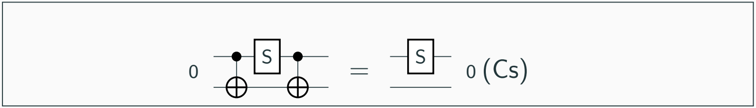

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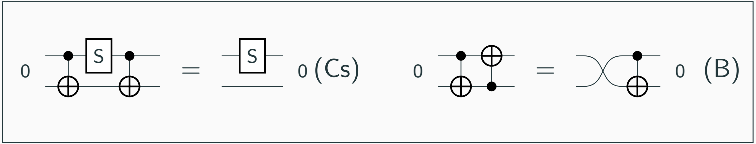

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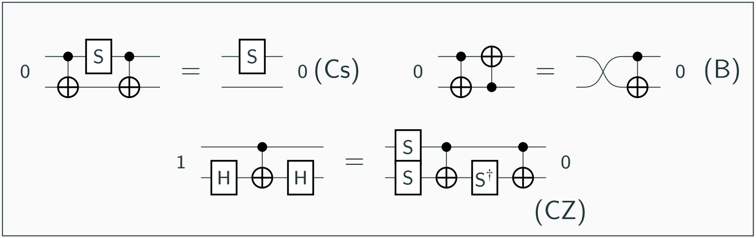

Comparison of Axiom Set Sizes

Fragment	Previous Theory	Our Theory
Qubit Clifford	15	8 (minimal for *n*-qubits)
Real Clifford	16	10 (minimal for *n*-qubits)
Qutrit Clifford	17	11 (minimal for 2-qutrits, conjectured for *n*)
Clifford+T	18	11 (minimal for 1-qubit, simplified for 2, no result for *n*)

Summary

- Graphical equational theories let us *reason* and *optimize* circuits visually.
- Our rule sets are *complete* (all true equivalences provable) and are *minimal* in the sense that we can't derive some equations from the others.
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